REES ALGEBRAS ON SMOOTH SCHEMES: INTEGRAL CLOSURE AND HIGHER DIFFERENTIAL OPERATORS.

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Abstract. Let V be a smooth scheme over a field k, and let $\{I_n, n \geq 0\}$ be a filtration of sheaves of ideals in $\mathcal{O}_V$, such that $I_0 = \mathcal{O}_V$, and $I_s \cdot I_t \subset I_{s+t}$. In such case $\bigoplus I_n$ is called a Rees algebra.

A Rees algebra is said to have differential structure if, for any two integers $N > n$ and any differential operator D of order n, $D(I_N) \subset I_{N-n}$. Any Rees algebra extends to a differential structure.

There are two extensions of Rees algebras: one defined by taking integral closures, and another by extending the algebra to a differential structure.

We study here some relations between these two extensions, with particular emphasis on the behavior of higher order differentials over arbitrary fields.

Contents

1. Introduction

A smooth ring R, of finite type over a field k, has a locally free sheaf of k-linear differential of order n, say $Diff^n(R)$, for each index $n \geq 0$.

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A graded subring of a polynomial ring over R, say $R[W]$, can be expressed as $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n$, where each I_n is an ideal in R. $\mathcal{G}$ is called the Rees algebra of the filtration $\{I_n\}_{n\geq 0}$. For example a Rees ring of an ideal I is of this type; in which case $I_n = I^n$. The integral closure of a Rees ring is also a Rees algebra (not necessarily a Rees ring of an ideal).

Taking integral closure of Rees algebras in $R[W]$ can be thought of as an operator, say $\mathcal{G} \subset \overline{\mathcal{G}}$.

The study of embedded singularities has motivated another kind of extension, linked to differential operators. In fact, from the point of view of singularities it is interesting to consider Rees algebras $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n$ with the additional property that for any operator $D \in Diff^n(R)$, and any index $N > n$, $D(I_N) \subset I_{N-n}$.

It is simple to show that for any $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n$, there is a smallest extension to one with this property, say $G(\mathcal{G})$. This defines a second operator, say $\mathcal{G} \subset G(\mathcal{G})$.

The objective of this paper is to study the interplay between both operators. In Th 6.12 it is shown that if two Rees algebras have the same integral closure, then their G-extensions also have the same integral closure. This shows a curious relation of differential operators with integral closures.

The techniques for this first part are developed in section 6. The key idea is to consider suitable weighted structures defined by coefficients of truncated Taylor development. In fact, it is in this context where the link of integral closure with differential operators arises.

This is a paper on commutative algebra; however it is motivated, and has applications, on the study of singularities over arbitrary fields ([16]), not treated in this presentation.

These extensions of graded algebras appear in [19] , and more recently in work of J. Wlodarczyk , and J. Kollár ([8], and [17]). But it is in the work of Hironaka [5],[6],[7] where the notion of differential structure is treated systematicaly in relation to the theory of infinitely close points. Our work is related with these last three papers, particularly with his "finite presentation theorem" in [6] . Here we do not make use of monoidal transformations, and hence of the theory of infinitely close points. The notion of restriction of differential structures, in section 5, appears already in the work of Hironaka.

Our interest and aim is on the case differential structures over fields of arbitrary characteristic. Differential operators of higher order have been a fundamental tool in various aspects of algebraic geometry over fields of characteristic $p > 0$.

We refer here to [15] for geometric applications of this work, and also for more details on the relation of our results with results of Hironaka, where these ideas where iniciated.

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2. Graded rings and Differential structures

2.1. Fix a noetherian ring B, and a sequence of ideals $\{I_k\}$, $k \geq 0$, which fulfill the conditions:

1) $I_0 = B$, and

2) $I_k \cdot I_s \subset I_{k+s}$.

This defines a graded subring $\mathcal{G} = \bigoplus_{k\geq 1} I_k \cdot W^k$ of the polynomial ring $B[W]$. We say that $\mathcal{G}$ is a *Rees algebra* only if this subring is a (noetherian) finitely generated B-algebra.

Remark 2.2. 1) Examples of Rees algebras are the Rees rings of an ideal $I \subset B$, where $I_k = I^k$ for each $k \geq 1$. In general we will not assume that a Rees algebra is generated in degree one.

2) Whenever $\bigoplus I_k \cdot W^k \subset (\subset B[W])$ is a Rees algebra, we may define a new Rees algebra $\bigoplus I'_k \cdot W^k$ by setting

$$I'_k = \sum_{r\geq k} I_r.$$

If $\bigoplus I_k \cdot W^k$ is generated by $\mathcal{F} = \{g_{n_i} W^{n_i}, 1 \leq i \leq m, n_i > 0\}$. Namely, if:

$$\bigoplus I_k \cdot W^k = R[\{g_{n_i} W^{n_i}\}_{g_{n_i} W^{n_i} \in \mathcal{F}}],$$

then $\bigoplus I'_k \cdot W^k$ is generated by the finite set $\{g_{n_i} W^{n'_i}, 1 \leq i \leq m, 1 \leq n'_i \leq n_i\}$,

Note that $I'_k \supset I'_{k+1}$, and that $\bigoplus I_k \cdot W^k \subset \bigoplus I'_k \cdot W^k$ is a finite extension. In fact, it suffices to check that given an element $g \in I_k$, then $g \cdot W^{k-1}$ is integral over $\bigoplus I_k \cdot W^k$. One can check that

$$g \in I_k \Rightarrow g^{k-1} \in I_{k(k-1)} \Rightarrow g^k \in I_{k(k-1)},$$

so $g \cdot W^{k-1}$ fulfills the equation $Z^k - (g^k \cdot W^{k(k-1)}) = 0$.

Up to integral closure we may assume that a Rees algebra has the additional condition:

$$I_k \supset I_{k+1}.$$

2.3. In what follows we define a Rees algebra, say $\bigoplus_{n\geq 0} I_n W^n$ in $B[W]$, by fixing a set of generators, say

$$\mathcal{F} = \{g_{n_i} W^{n_i} / n_i > 0, 1 \leq i \leq m\}.$$

So if $f \in I_n$, then

$$f = F_n(g_{n_1}, \dots, g_{n_m}),$$

where $F_n(Y_1, \dots, Y_m)$ is a weighted homogeneous polynomial in m variables, and each Y_j has weight n_j.

For a fixed positive integer N, $B[W^N] \subset B[W]$ is a finite extension of graded rings. Furthermore, $\bigoplus_{k\geq 0} I_{kN} W^{kN}$ is a Rees algebra in $B[W^N]$, and $\bigoplus_{k\geq 0} I_{kN} W^{kN} \subset \bigoplus_{n\geq 0} I_n W^n$ is also finite.

Note that if N is a common multiple of all integers n_i, $1 \leq i \leq m$, then

$$\bigoplus_{k\geq 0} I_N^k W^{kN} \subset \bigoplus_{n\geq 0} I_n W^n$$

is an integral extension, where the left hand side is the Rees ring of I_N (in $B[W^N]$). So Rees algebras are finite extensions of Rees rings.

If a Rees algebra $\bigoplus_{n\geq 0} I_n W^n$ in $B[W]$ is the Rees ring of I_1, then the integral closure in $B[W]$ is $\bigoplus_{n\geq 0} \overline{I}_n W^n$, where each $\overline{I}_n$ is the integral closure of the ideal I_n. This is a Rees algebra, and not necessarily the Rees ring of the ideal $\overline{I}_1$.

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2.4. Let B be a normal excellent ring, and let

$$Spec(B) \xleftarrow{\pi} X$$

be a proper birational morphism, then $I \subset \pi_*(I\mathcal{O}_X) \subset \overline{I}$, where $\overline{I}$ denotes the integral closure of I in B. Moreover, if π is the normalization of the blow-up at I, then $I\mathcal{O}_X$ is an invertible sheaf of ideals, and

$$\overline{I} = \pi_*(I\mathcal{O}_X).$$

Assume that the normal ring B is of finite type over a field k. If B is a one dimensional normal domain, any ideal is invertible and integrally closed. We add the following well known result for self-containment (se [5], p.54 or [9] p. 100).

Lemma 2.5. *Let I, J be two ideals in a normal domain B, which is finitely generated over a field k. Then $\overline{I} = \overline{J}$ if and only if $I\mathcal{O}_W = J\mathcal{O}_W$, for any morphism of k-schemes $W \to Spec(B)$, with W of dimension one, regular and of finite type over k.*

Let $x \in W$ map to $y \in W$, then $\mathcal{O}_{W,x}$ is a valuation ring that dominates $\mathcal{O}_{Spec(B),y}$. So if $\overline{I} = \overline{J}$, then $I\mathcal{O}_W = J\mathcal{O}_W$. In fact, for any morphism $B \to A$, where A is a valuation ring, $IA = \overline{I}A$.

Assume that this condition holds for any morphism from a regular one dimensional scheme W. We claim now that both ideals have the same integral closure in B.

Let $Spec(B) \xleftarrow{\pi} X$ be the normalized blow up at I, and let $\{H_1, \dots, H_s\}$ be the irreducible components of the closed set defined by the invertible sheaf of ideals $I\mathcal{O}_X$. Here each H_i is an irreducible hypersurfaces in X. Let $h_i \in X$ denote the generic point of H_i. There are positive integers a_i, so that $I\mathcal{O}_X$ can be characterized as the sheaf of functions vanishing along H_i with order at least a_i (i.e. with order at least a_i at the valuation rings $\mathcal{O}_{X,h_i}$).

Claim: The sheaf of ideals $J\mathcal{O}_X$ also has order a_i at $\mathcal{O}_{X,h_i}$.

If the claim holds, $J\mathcal{O}_X \subset I\mathcal{O}_X$, and

$$J \subset \pi_*(J\mathcal{O}_X) \subset \pi_*(I\mathcal{O}_X) = \overline{I}.$$

In particular $\overline{J} \subset \overline{I}$. A similar argument would lead to the other inclusion.

In order to prove the claim we choose a closed point $x \in H_i$ so that:

1) $\mathcal{O}_{X,x}$ is regular,

2) $x \in H_i - \cap_{j \neq i} H_j$,

3) H_i is regular at x, and

4) $J\mathcal{O}_{X,x}$ is a p-primary ideal, for $p = I(H_i)_x$.

Since any sheaf of ideals has only finitely many p-primary components, such choice of x is possible.

Let $\{x_1, \dots x_{d-1}, x_d\}$ be a regular system of parameters at $\mathcal{O}_{X,x}$ such that $p = I(H_i)_x = x_d\mathcal{O}_{X,x}$, and let W be the closure of the irreducible curve defined locally by $< x_1, \dots, x_{d-1} >$. So W is one dimensional, and regular locally at x. We may assume that W is regular after applying quadratic transformations which do not affect the local ring $\mathcal{O}_{W,x}$. By construction $I\mathcal{O}_{W,x}$ has order a_i, by hypothesis the same holds for $J\mathcal{O}_{W,x}$. This proves the claim.

2.6. Let $B = S[X]$ be a polynomial ring, and let $Tay : B \to B[U]$ be the S-algebra homomorphism defined by setting $Tay(X) = X + U$. For any $f(X) \in B$ set

$$Tay(f(X)) = \sum_{\alpha \geq 0} \Delta^{\alpha}(f(X))U^{\alpha}.$$

The operators Δ^{α} are S-differential operators (S linear). Furthermore, for any positive integer N, the set $\{\Delta^{\alpha}, 0 \leq \alpha \leq N\}$ is a basis of the B-module of S-differential operators on B, of order $\leq N$.

Definition 2.7. Let $B = S[X]$ be a polynomial ring over a noetherian ring S. A Rees algebra

$$\bigoplus I_k \cdot W^k \subset B[W]$$

is a differential structure, say Diff-structure, relative to S, when:

i) $I_k \supset I_{k+1}$ for any $k \geq 0$.

ii) For any $n > 0$ and $f \in I_n$, and for any index $0 \leq j \leq n$ and any S-differential operator of order $\leq j$, say D_j:

$$D_j(f) \in I_{n-j}.$$

Remark 2.8. Let $Diff_S^N(B)$ denote the module of S-differential operators of order at most N. Then

$$Diff_S^N(B) \subset Diff_S^{N+1}(B) \subset \dots$$

For this reason it is natural to require condition (i) in our previous definition. Note also that 2.6 asserts that (ii) can be reformulated as:

ii') For any $n > 0$ and $f \in I_n$, and for any index $0 \leq \alpha \leq n$:

$$\Delta^{\alpha}(f) \in I_{n-\alpha},$$

In fact, (i)+(ii) is equivalent to (i)+(ii'):

Theorem 2.9. *Fix $B = S[X]$ as before, and a finite set $\mathcal{F} = \{g_{n_i}W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, with the following properties:*

a) For any $1 \leq i \leq m$, and any n'_i, $0 < n'_i \leq n_i$:

$$g_{n_i}W^{n'_i} \in \mathcal{F}.$$

b) For any $1 \leq i \leq m$, and for any index $0 \leq \alpha < n_i$:

$$\Delta^{\alpha}(g_n)W^{n_i - \alpha} \in \mathcal{F}.$$

Then the B subalgebra of $B[W]$, generated by $\mathcal{F}$ over the ring B, has Diff-structure relative to S.

Proof. Condition (i) in Def 2.7 is by 2.2, 2).

Let $I_N W^N$ be the homogeneous component of degree N of the B subalgebra generated by $\mathcal{F}$. We prove that for any $h \in I_N$, and any $0 \leq \alpha \leq N$, $\Delta^{\alpha}(h) \in I_{N-\alpha}$.

The ideal $I_N \subset B$ is generated by all elements of the form

$$H_N = g_{n_{i_1}} \cdot g_{n_{i_2}} \cdots g_{n_{i_p}} \qquad n_{i_1} + n_{i_2} + \cdots n_{i_p} = N, \tag{2.9.1}$$

with the $g_{n_{i_i}} W^{n_{i_i}} \in \mathcal{F}$ not necessarily different.

Since the operators Δ^α are linear, it suffices to prove that $\Delta^\alpha(a \cdot H_N) \in I_{N-\alpha}$, for $a \in B$, H_N as in 2.9.1, and $0 \leq \alpha \leq N$. We proceed in two steps, by proving:

1) $\Delta^\alpha(H_N) \in I_{N-\alpha}$.

2) $\Delta^\alpha(a \cdot H_N) \in I_{N-\alpha}$.

We first prove 1). Set $Tay : B = S[X] \to B[U]$, as in 2.6. Consider, for any element $g_{n_{i_l}} W^{n_{i_l}} \in \mathcal{F}$,

$$Tay(g_{n_{i_l}}) = \sum_{\beta \geq 0} \Delta^\beta(g_{n_{i_l}}) U^\beta \in B[U].$$

Hypothesis (b) states that for each index $0 \leq \beta < n_{i_l}$, $\Delta^\beta(g_{n_{i_l}}) W^{n_{i_l} - \beta} \in \mathcal{F}$.

On the one hand

$$Tay(H_N) = \sum_{\alpha \geq 0} \Delta^\alpha(H_N) U^\alpha,$$

and, on the other hand

$$Tay(H_N) = Tay(g_{n_{i_1}}) \cdot Tay(g_{n_{i_2}}) \cdots Tay(g_{n_{i_p}})$$

in $B[U]$. This shows that for a fixed α ($0 \leq \alpha \leq N$), $\Delta^\alpha(H_N)$ is a sum of elements of the form:

$$\Delta^{\beta_1}(g_{n_{i_1}}) \cdot \Delta^{\beta_2}(g_{n_{i_2}}) \cdots \Delta^{\beta_p}(g_{n_{i_p}}), \qquad \sum_{1 \leq s \leq p} \beta_s = \alpha.$$

So it suffices to show that each of these summands is in $I_{N-\alpha}$.

Note here that

$$\sum_{1 \leq s \leq p} (n_{i_s} - \beta_s) = N - \alpha,$$

and that some of the integers $n_{i_s} - \beta_s$ might be zero or negative. Set

$$G = \{r, 1 \leq r \leq p, \text{ and } n_{i_r} - \beta_r > 0\}.$$

So

$$N - \alpha = \sum_{1 \leq s \leq p} (n_{i_s} - \beta_s) \leq \sum_{r \in G} (n_{i_r} - \beta_r) = M.$$

Hypothesis (b) ensures that $\Delta^{\beta_r}(g_{n_{i_r}}) \in I_{n_{i_r} - \beta_r}$ for every index $r \in G$, in particular:

$$\Delta^{\beta_1}(g_{n_{i_1}}) \cdot \Delta^{\beta_2}(g_{n_{i_2}}) \cdots \Delta^{\beta_p}(g_{n_{i_p}}) \in I_M.$$

Finally, since $M \geq N - \alpha$, $I_M \subset I_{N-\alpha}$, and this proves Case 1).

For Case 2), fix $0 \leq \alpha \leq N$. We claim that $\Delta^\alpha(a \cdot H_N) \in I_{N-\alpha}$, for $a \in B$ and H_N as in 2.9.1. At the ring $B[U]$,

$$Tay(a \cdot H_N) = \sum_{\alpha \geq 0} \Delta^\alpha(a \cdot H_N) U^\alpha,$$

and, on the other hand

$$Tay(a \cdot H_N) = Tay(a) \cdot Tay(H_N).$$

This shows that $\Delta^\alpha(a \cdot H_N)$ is a sum of terms of the form $\Delta^{\alpha_1}(a) \cdot \Delta^{\alpha_2}(H_N)$,$\alpha_i \geq 0$, and $\alpha_1 + \alpha_2 = \alpha$. In particular $\alpha_2 \leq \alpha$; and by Case 1), $\Delta^{\alpha_2}(H_N) \in I_{N-\alpha_2}$. On the other hand $N - \alpha_2 \geq N - \alpha$, so $\Delta^{\alpha_2}(H_N) \in I_{N-\alpha}$, and hence $\Delta^\alpha(a \cdot H_N) \in I_{N-\alpha}$. $\square$

Corollary 2.10. *The Rees algebra in $B[W]$, generated over B by*

$$\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\},$$

extends to a smallest Diff-structure, which is generated by the finite set

$$\mathcal{F}' = \{\Delta^\alpha(g_n) W^{n_i - \alpha} / g_{n_i} W^{n_i'} \in \mathcal{F}, \text{ and } 0 \leq \alpha < n_i\}.$$

3. Differential structures on smooth schemes.

3.1. A sequence of coherent ideals on a scheme Z, say $\{I_n\}_{n \in \mathbb{N}}$, such that $I_0 = \mathcal{O}_Z$, and $I_k \cdot I_s \subset I_{k+s}$, defines a graded sheaf of algebras $\bigoplus_{n \geq 0} I_n \cdot W^n \subset \mathcal{O}_Z[W]$.

We say that this algebra is a Rees algebra if there is an open covering of Z by affine open sets $\{U_i\}$, so that

$$\bigoplus_n I_n(U_i) W^n \subset \mathcal{O}_Z(U_i)[W]$$

is a finitely generated $\mathcal{O}_Z(U_i)$-algebra.

In what follows Z will denote a smooth scheme of a field k, and $Diff_k^r(Z)$, or simply $Diff_k^r$, the locally free sheaf of k-linear differential operators of order at most r.

Definition 3.2. We say that a Rees algebra defined by $\{I_n\}_{n \in \mathbb{N}}$ is a Diff-structure relative to the field k, if:

i) $I_n \supset I_{n+1}$.

ii) There is open covering of Z by affine open sets $\{U_i\}$, and for any $D \in Diff^{(r)}(U_i)$, and any $h \in I_n(U_i)$, then $D(h) \in I_{n-r}(U_i)$, provided $n \geq r$.

Due to the local nature of the definition, we reformulate it in terms of smooth k-algebras.

Definition 3.3. In what follows R will denote a smooth algebra over a field, or a localization of such algebra on a closed point (a regular local ring). A Rees algebra is defined by a sequences of ideals $\{I_k\}_{k \in \mathbb{N}}$ such that:

1) $I_0 = R$, and $I_k \cdot I_s \subset I_{k+s}$.

2) $\bigoplus I_k W^k$ is a finitely generated R-algebra.

We shall say that the Rees algebra has Diff-structure relative to k, if

3) $I_n \supset I_{n+1}$, and

4) given $D \in Diff_k^{(r)}(R)$, then $D(I_n) \subset I_{n-r}$.

We now show that any Rees algebra extends to a smallest Diff-structure (i.e. included in any other Diff-structure containing it).

Theorem 3.4. *Assume that $\mathcal{G} = \bigoplus I_k \cdot W^k$ is a Rees algebra over a smooth scheme Z. Then there is a natural and smallest extension of it, say $\mathcal{G} \subset G(\mathcal{G})$, where $G(\mathcal{G})$ is a Diff-structure relative to the field k.*

Proof. The problem is local, so we will assume that R is the local ring at a closed point. We show that a finitely generated subalgebra of $R[W]$ extends, by successive applications of differential operators, to a finitely generated algebra.

We will argue in steps. Assume that the local ring R is of dimension 1, and let x denote a parameter. Set $Tay : \hat{R} \to \hat{R}[[U]]$ the k-algebra morphism at the completion defined by setting $Tay(x) = x+U$. Here $\hat{R} = k'[[x]]$ is a ring of formal power series over a finite extension k' of k,

$$Tay(f) = \sum \Delta^r(f(x))U^r,$$

and each

$$\Delta^r : k'[[x]] \to k'[[x]]$$

maps R into R, defining

$$\Delta^r : R \to R.$$

So $Tay : \hat{R} \to \hat{R}[[U]]$ induces by restriction $Tay : R \to R[[U]]$.

For any $f \in R$ set

$$Tay(f) = \sum_{r \geq 0} \Delta^r(f)U^r (\in R[[U]]).$$

The operators Δ^r are a basis of the k-linear differential operators on R.

The same argument used in Theorem 2.9 shows that if $\bigoplus I_k \cdot W^k$ is generated by $\mathcal{F} = \{g_{n_i}W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, then

$$\mathcal{F}' = \{\Delta^r(g_n)W^{n_i'-r}/g_{n_i}W^{n_i} \in \mathcal{F}, \text{ and } 0 \leq r < n_i' \leq n_i\}$$

generates the smallest extension to a Diff-structure.

Let now R be a localization of an arbitrary smooth algebra at a closed point, and fix a regular system of parameters $\{x_1, \dots, x_n\}$. Define

$$Tay : \hat{R} \to \hat{R}[[U_1, \dots, U_n]]$$

as the continuous morphisms of algebras defined by setting $Tay(x_i) = x_i + U_i$. So for any $h \in \hat{R}$ set:

$$Tay(h) = \sum_{\alpha \in (\mathbb{N})^n} \Delta^\alpha(h)U^\alpha.$$

This morphism defines, by restriction, $Tay : R \to R[[U_1, \dots, U_n]]$, and we set

$$Tay(g) = \sum_{\alpha \in (\mathbb{N})^n} \Delta^\alpha(g)U^\alpha.$$

$\{\Delta^\alpha/\alpha \in (\mathbb{N})^n, 0 \leq |\alpha| \leq n\}$ is a basis of the free R-module $Diff^n(R)$, and in order to show that a Rees algebra $\bigoplus I_k \cdot W^k$ has Diff-structure, it suffices to check that given $g \in I_m$:

(3.4.1) $$\Delta^\alpha(g) \in I_{m-|\alpha|}.$$

Note that
$$\Delta^\alpha \Delta^{\alpha'} = \Delta^{\alpha'} \Delta^\alpha.$$

Define, for each index i_0, $1 \leq i_0 \leq n$:
$$Tay_{i_0} : R \to R[[U_{i_0}]],$$
$Tay_i(x_j) = x_j$ and $Tay_i(x_{i_0}) = x_{i_0} + U_{i_0}$. So
$$Tay_{i_0}(g) = \sum_{\alpha \in \mathbb{N}} \Delta^\alpha_{i_0}(g) U^\alpha,$$
is defined in terms of the partial differential operators $\Delta^\alpha_{i_0}$.

For any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N})^n$:
$$\Delta^\alpha = \Delta_1^{\alpha_1} \cdots \Delta_n^{\alpha_n},$$
is a composition of partial operators defined above. And $\bigoplus I_k \cdot W^k$ has Diff-structure if the requirement in (3.4.1) holds for each of these partial differential operators.

So again, the arguments in Theorem 2.9 ensure that if $\bigoplus I_k \cdot W^k$ is generated by
$$\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\},$$
then

(3.4.2) $$\mathcal{F}' = \{\Delta^\alpha(g_n) W^{n'_i - \alpha} / g_{n_i} W^{n_i} \in \mathcal{F}, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N})^n, \text{ and } 0 \leq |\alpha| < n'_i \leq n_i\}$$

generates the smallest extension of $\bigoplus I_k \cdot W^k$ with Diff-structure relative to the field k.

□

Corollary 3.5. *Given inclusions of Rees algebras, say*
$$\mathcal{G} = \bigoplus I_n W^n \subset \mathcal{G}' = \bigoplus I'_n W^n \subset G(\mathcal{G}) = \bigoplus I''_n W^n,$$
where $G(\mathcal{G})$ is the Diff-structure spanned by $\mathcal{G}$, then $G(\mathcal{G})$ is also the Diff-structure spanned by $\mathcal{G}'$.

3.6. Fix now a smooth morphism of smooth schemes, say $Z \to Z'$. Let $Diff^r_{Z'}(Z)$, or simply $Diff^r_{Z'}$ denote the locally free sheaf of relative differential operators of order r.

We say that the Rees algebra $\bigoplus I_k \cdot W^k$ over Z (3.1) has Diff-structure relative to Z', if conditions in Def 3.2 hold, where we now require that $D \in Diff^{(r)}_{Z'}(U_i)$ in (ii).

Since $Diff^r_{Z'}(Z) \subset Diff^r_k(Z)$ it follows that any Diff-structure relative to k is also relative to Z'.

Theorem 3.4 has a natural formulation for the case of Diff-structures relative to Z'.

Given an ideal $I \subset \mathcal{O}_Z$, and a smooth morphism $Z \to Z'$, we define an extension of ideals $I \subset Diff^r_{Z'}(I)$,

$$Diff^r_{Z'}(I)(U) = \langle D(f)/f \in I(U), D \in Diff^r_{Z'}(U)\rangle$$

for each open U in Z.

Since $Diff^r_{Z'} \subset Diff^{r+1}_{Z'}$, clearly $Diff^r_{Z'}(I) \subset Diff^{r+1}_{Z'}(I)$ for $r \geq 0$.

Note finally that a Rees algebra $\bigoplus I_k \cdot W^k$ over Z (3.1) has Diff-structure relative to Z', if and only if, for any positive integers $r \leq n$, $Diff^r_{Z'}(I_n) \subset I_{n-r}$. In particular, for $Z' = Spec(k)$, condition ii) in Def 3.2 can be reformulated as:

ii') $Diff^r_k(I_n) \subset I_{n-r}$.

4. Differential structures and singular locus.

4.1. The notion Diff-structure relative to a field k, on a smooth k-scheme Z, is closely related to the notion of *order* at the local regular rings of Z. Recall that the order of a non-zero ideal I at a local regular ring (R, M) is the biggest integer b such that $I \subset M^b$.

If $I \subset \mathcal{O}_Z$ is a sheaf of ideals, $V(Diff^{b-1}_k(I))$ is the closed set of points of Z where the ideal has order at least b. We analyze this fact locally at a closed point x.

Let $\{x_1, \ldots, x_n\}$ be a regular system of parameters at $\mathcal{O}_{Z,x}$, and consider the differential operators Δ^α, defined on $\mathcal{O}_{Z,x}$ in terms of these parameters, as in the Theorem 3.4. So at x,

$$(Diff^{b-1}_k(I))_x = \langle \Delta^\alpha(f)/f \in I, 0 \leq |\alpha| \leq b-1\rangle.$$

One can now check at $\mathcal{O}_{Z,x}$, or at the ring of formal power series $\hat{\mathcal{O}}_{Z,x}$, that $Diff^{b-1}_k(I)$ is a proper ideal if and only if I has order at least b at the local ring.

The operators Δ^α are defined globally at a suitable neighborhood U of x. So if $\bigoplus I_n \cdot W^n \subset O_Z[W]$ is a Diff-structure relative to the field k and $x \in Z$ is a closed point, the Diff-structure $\bigoplus (I_n)_x \cdot W^n \subset O_{Z,x}[W]$ is properly included in $\mathcal{O}_{Z,x}[W]$, if and only, for each index $k \in \mathbb{N}$, the ideal $(I_k)_x$ has order at least k at the local regular ring $\mathcal{O}_{Z,x}$.

Definition 4.2. The *singular locus* of a Rees algebra $\mathcal{G} = \bigoplus I_n \cdot W^n \subset O_Z[W]$, will be

$$Sing(\mathcal{G}) = \cap_{r\geq 0} V(Diff^{r-1}_k(I_r))(\subset Z).$$

It is the set of points $x \in Z$ for which all $(I_r)_x$ have order at least r (at $\mathcal{O}_{Z,x}$).

Remark 4.3. Assume that $f \in (I_r)_x$ has order r at $\mathcal{O}_{Z,x}$. Then, locally at x, $Sing(\mathcal{G})$ is included in the set of points of multiplicity r (or say, r-fold points) of the hypersurface $V(\langle f\rangle)$.

In fact $Diff^{r-1}_k(f) \subset Diff^{r-1}_k(I_r)$, and the closed set defined by the first ideal is that of points of multiplicity r.

Proposition 4.4. *1) If $\mathcal{G} = \bigoplus I_n W^n$ and $\mathcal{G}' = \bigoplus I'_n W^n$ are Rees algebras with the same integral closure (e.g. if $\mathcal{G} \subset \mathcal{G}'$ is a finite extension), then*

$$Sing(\mathcal{G}) = Sing(\mathcal{G}').$$

2) If $\mathcal{G}$ is a Rees algebra generated over $\mathcal{O}_Z$ by $\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, then

$$Sing(\mathcal{G}) = \cap V(Diff^{n_i}(\langle g_i\rangle)).$$

3) Let $\mathcal{G}'' = \bigoplus I''_n \cdot W^n$ be the extension of $\mathcal{G}$ to a Diff-structure relative to k, as defined in Theorem 3.4, then $Sing(\mathcal{G}) = Sing(\mathcal{G}'')$.

4) For any Diff-structure $\mathcal{G}'' = \bigoplus I''_n \cdot W^n$, $Sing(\mathcal{G}'') = V(I''_1)$.

5) Let $\mathcal{G}'' = \bigoplus I''_n \cdot W^n$ be a Diff-structure. For any positive integer r, $Sing(\mathcal{G}'') = V(I''_r)$.

Proof. 1) The argument in 2.3 shows that there is an index N, so that $\mathcal{G}$ is finite over the subring $\bigoplus I_N^k W^{Nk}$, and $\mathcal{G}'$ is finite over $\bigoplus I'^k_N W^{Nk}$. And furthermore, I_N and I'_N have the same integral closure. In these conditions $Sing(\mathcal{G})$ is the set of points $x \in Z$ such that I_N has order at least N at $\mathcal{O}_{Z,x}$, and similarly, $Sing(\mathcal{G}')$ is the set of points $x \in Z$ such that I'_N has order at least N. Finally, the claim follows from the fact that the order of an ideal, at a local regular ring, is the same as the order of its integral closure ([20], Appendix 3).

2) We have formulated 2) with a global condition on Z, however this is always the case locally. In fact, there is a covering of Z by affine open sets, so that the restriction of $\mathcal{G}$ is generated by finitely many elements. Let U be such open set, so $\mathcal{G}(U) = \bigoplus I_k(U) \cdot W^k$ is generated by $\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, $g_{n_i} \in \mathcal{O}(U)$.

The claim is that $y \in Sing(\mathcal{G}) \cap U$ if and only if the order of g_{n_i} at $\mathcal{O}_{Z,y}$ is at least n_i, for $1 \leq i \leq m$.

The condition is clearly necessary. Conversely, if $\mathcal{G} = \bigoplus I_n = \mathcal{O}_U[\{g_i W^{n_i}\}_{g_i W^{n_i} \in \mathcal{F}}]$, and each g_{n_i} has order at at least n_i at $\mathcal{O}_{Z,y}$, then I_n (generated by weighted homogeneous expressions on the g_i's) has order at least n at $\mathcal{O}_{Z,y}$.

3) We argue as in 2), here we may also assume that there is $x \in U$, a regular system of parameters $\{x_1, \dots, x_n\}$ at x, and differential operators Δ^α as in the Theorem 3.4, defined globally at U.

The Diff-structure $\mathcal{G}''$ in the Theorem 3.4, is a finite extension of the Rees algebra defined by

$$\mathcal{F}' = \{\Delta^\alpha(g_n) W^{n_i - \alpha} / g_{n_i} W^{n_i} \in \mathcal{F}, \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{N})^n, \text{ and } 0 \leq |\alpha| < n_i\}.$$

Note finally that if the order of g_{n_i} at a local ring is $\geq n_i$, then the order of $\Delta^\alpha(g_n)$ is $\geq n_i - |\alpha|$.

4) The inclusion $Sing(\mathcal{G}'') \subset V(I''_1)$ holds, by definition, for any Rees algebra. On the other hand, the hypothesis ensures that $Diff^{r-1}(I''_r) \subset I''_1$, so $Sing(\mathcal{G}'') \supset V(I''_1)$.

5) Follows from 4). □

5. On restrictions of differential structures.

Proposition 5.1. *Let $\mathcal{G} = \bigoplus I_k \cdot W^k$ be a Diff-structure on a smooth scheme V defined by ideals $I_k \subset \mathcal{O}_V$.*

A) If $V' \subset V$ is a smooth subscheme, the restriction of $\mathcal{G}$ to V', say

$$\mathcal{G}' = \bigoplus I_k \mathcal{O}_{V'} \cdot W^k,$$

is a Diff-structure on V'.

B) If $V'' \to V$ is a smooth morphism, then the natural extension, say

$$\mathcal{G}'' = \bigoplus I_k \mathcal{O}_{V''} \cdot W^k,$$

is a Diff-structure on V''.

Proof. It is clear that both $\mathcal{G}'$ and $\mathcal{G}''$ are Rees algebras (3.1). We will show that conditions (i) and (ii) in Definition 3.2 hold.

It suffices to prove both results locally at closed points, say $x \in Sing(\mathcal{G})$. Set $\mathcal{G}_x = \bigoplus I_k \cdot W^k$ where now each I_n is an ideal in $\mathcal{O}_{V,x}$. We may also replace the local ring by its completion.

A) Fix a closed point $x \in V' \subset V$ and a local regular system of parameters, say

$$\{x_1, \dots, x_h, x_{h+1}, \dots x_d\}$$

at $\mathcal{O}_{V,x}$, such that V' is locally defined by the ideal $< x_1, \dots, x_h >$. Set

$$\hat{\mathcal{O}}_{V,x} = k'[[x_1, \dots, x_h, x_{h+1}, \dots x_d]],$$

where k' is a finite extension of k. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$,

$$\Delta^{\alpha} = \Delta^{\alpha^{(1)}} \Delta^{\alpha^{(2)}};$$

where $\alpha^{(1)} = (\alpha_1, \dots, \alpha_h) \in \mathbb{N}^h$, and $\alpha^{(2)} = (\alpha_{h+1}, \dots, \alpha_d) \in \mathbb{N}^{d-h}$.

Express an element $f_n \in I_n$ as

$$f_n = \sum_{\alpha^{(1)} \in (\mathbb{N})^h} x_1^{\alpha_1} \cdots x_h^{\alpha_h} a_{\alpha^{(1)}},$$

$a_{\alpha^{(1)}} \in k'[[x_{h+1}, \dots x_d]]$.

If $|\alpha^{(1)}| = \alpha_1 + \cdots + \alpha_h \leq n$, then $a_{\alpha^{(1)}} W^{n-|\alpha^{(1)}|}$ is the class of $\Delta^{\alpha^{(1)}}(f_n) \cdot W^{n-|\alpha^{(1)}|}$ in $\hat{\mathcal{O}}_{V',x}[W]$. So it is an element in the restricted algebra. Similarly, if $|\alpha^{(1)}| + |\beta^{(2)}| \leq n$,

$$\Delta^{\beta^{(2)}} a_{\alpha^{(1)}} W^{n-|\alpha^{(1)}|-|\beta^{(2)}|}$$

is the class of the element $(\Delta^{\beta^{(2)}} \Delta^{\alpha^{(1)}})(f_n) \cdot W^{n-|\alpha^{(1)}|-|\beta^{(2)}|}$ in $\hat{\mathcal{O}}_{V',x}[W]$.

For each index $m \geq 1$, $I_m \mathcal{O}_{V'} \cdot W^m$ is defined by the coefficient $a_0 W^m$ $(0 \in (\mathbb{N})^h)$, for each $f_m W^m \in I_m W^m$. Conditions (i) and (ii) in 3.2 are now easy to check.

For our further discussion we point out that $I_m \mathcal{O}_{V'} \cdot W^m$ also contains all coefficients $a_{\alpha^{(1)}} W^{n-|\alpha^{(1)}|}$ of $fW^n \in I^n W$, and $n - |\alpha^{(1)}| = m$.

B) Fix a point $x' \in V''$ mapping to $x \in V$. The completion of $\mathcal{O}_{V'',x'}$ contains that of $\mathcal{O}_{V,x}$, say

$$\hat{\mathcal{O}}_{V,x} = k'[[x_1, \dots, x_d]] \subset \hat{\mathcal{O}}_{V'',x'} = k'[[x_1, \dots, x_d, x_{d+1}, \dots x_e]].$$

Each ideal I_n in $k'[[x_1, \dots, x_d]]$ extends to $I_n \cdot k'[[x_1, \dots, x_d, x_{d+1}, \dots x_e]]$; and the claim is that the extended algebra has Diff-structure. The statement follows easily in this case, for example by formula (3.4.2), which expresses generators of the Diff-structure in terms of generators of the Rees algebra.

Definition 5.2. Fix $\mathcal{G} = \bigoplus I_k \cdot W^k$, a Rees algebra on V, and let $V \longleftarrow V'$ be a morphism of smooth schemes. We define the *total transform* of $\mathcal{G}$ to be

$$\pi^{-1}(\mathcal{G}) = \bigoplus I_k \mathcal{O}_{V'} \cdot W^k.$$

Namely the Rees algebra defined by the total transforms of the ideals I_n, $n \geq 0$.

Note that the *restriction* in A) and the *natural extension* in B), are particular examples of total transforms.

Lemma 5.3. *Let* $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_V[W])$ *be a* Rees algebra *generated by a finite set* $\mathcal{F} = \{g_{N_1} W^{N_1}, \dots, g_{N_s} W^{N_s}\}$, *and let* $V \longleftarrow V'$ *be a morphism of smooth schemes. Then* $\pi^{-1}(\mathcal{G})$ *is also generated by* $\mathcal{F}$.

Proof. Since any element of I_M is a weighted homogeneous polynomial of degree M in elements of $\mathcal{F}$, the total transform of the ideal is also generated by elements that are weighted homogeneous in the same set $\mathcal{F}$. $\square$

In particular:
A) the restriction of $\mathcal{G}$ to $V' (\subset V)$ is generated by $\{\overline{g}_{N_1} W^{N_1}, \dots, \overline{g}_{N_s} W^{N_1}\}$, where each $\overline{g}_{N_i}$ is the restriction of g_{N_i} to V'.
B) If $V'' \to V$ is a smooth morphism, the total transform of $\bigoplus I_k \cdot W^k$ to V'' is generated by $\{g_{N_1} W^{N_1}, \dots, g_{N_s} W^{N_s}\}$.

Theorem 5.4. *Let* $V' \xrightarrow{\pi} V$ *be a morphism of smooth schemes, then:*
i) if $\mathcal{G}$ *is a Diff-structure on* V, *the total transform* $\pi^{-1}(\mathcal{G})$ *is a Diff-structure on* V'.
ii) $Sing(\pi^{-1}(\mathcal{G})) = \pi^{-1}(Sing(\mathcal{G}))$.

Proof. Since $V' \xrightarrow{\pi} V$ is of finite type, it can be expressed locally in the form $V' \subset V'' \xrightarrow{\beta} V$, where β is smooth. So Prop 5.1 proves (i).

Fix a closed point $x \in Sing(\pi^{-1}(\mathcal{G}))$. Since $Sing(\mathcal{G}) = V(I_n)$ for all $n \geq 1$ (4.4), it follows that $\pi(x) \in Sing(\mathcal{G})$. On the other hand, if $\pi(x) \in Sing(\mathcal{G})$, the order of I_n is at least n at $\mathcal{O}_{V,\pi(x)}$, for each $n \geq 1$. So the same holds at $\mathcal{O}_{V',x}$. This proves (ii).

6. On differential structures and integral closures.

6.1. Fix a Rees algebra $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_V[W])$, and a point $x \in Sing(\mathcal{G})$. Let Z be a smooth subscheme containing x, and let $x_1, \dots, x_h$ be part of a regular system of parameters at $\mathcal{O}_{V,x}$ so that $< x_1, \dots, x_h >$ is the ideal defining Z locally at x. We will now define a graded algebra over the completion, namely in $\hat{\mathcal{O}}_{Z,x}[W]$.

This new graded algebra will be defined in terms of the (local) inclusion $Z \subset V$, and the retraction $V \to Z$ defined locally at x (see 6.3).

Extend $\{x_1, \dots, x_h\}$ to a regular system of parameters, say $\{x_1, \dots, x_h, x_{h+1}, \dots x_d\}$, of $\mathcal{O}_{V,x}$. $\hat{\mathcal{O}}_{V,x}$ is a ring of formal power series, say $k'[[x_1, \dots, x_h, x_{h+1}, \dots x_d]]$, and $\hat{\mathcal{O}}_{V',x}$ is $k'[[x_{h+1}, \dots x_d]]$. The local retraction is defined by $k'[[x_{h+1}, \dots x_d]] \subset k'[[x_1, \dots, x_h, x_{h+1}, \dots x_d]]$.

Set, as usual, $\mathcal{G}_x = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_{V,x}[W])$, that also extends to a Rees algebra over $\hat{\mathcal{O}}_{V,x}$. Express an element $f_n \in I_n$ as

$$f_n = \sum_{\alpha^{(1)} \in (\mathbb{N})^h} x_1^{\alpha_1} \cdots x_h^{\alpha_h} a_{\alpha^{(1)}}, \qquad a_{\alpha^{(1)}} \in k'[[x_{h+1}, \dots x_d]].$$

For any such $f_n W^n$, consider the set $\{a_{\alpha^{(1)}} \cdot W^{n-|\alpha^{(1)}|}, 0 \leq |\alpha^{(1)}| < n\}$, which we call the *coefficients of* $f_n W^n$. So the coefficients of $f_n W^n$ is a finite set, defined in terms of a regular system of parameters, and the weight of each coefficient depends on the index n.

Claim: As $f_n W^n$ varies on the Rees algebra $\mathcal{G}_x$, the coefficients of $f_n W^n$ generate a Rees algebra, say $\mathrm{Coeff}(\mathcal{G})_x$, in $k'[[x_{h+1}, \dots x_d]][W]$.

The claim here is that the graded algebra $\mathrm{Coeff}(\mathcal{G})_x$ is a finitely generated subalgebra of $k'[[x_{h+1}, \dots x_d]][W]$.

Assume that $\mathcal{F} = \{g_{N_1} W^{N_1}, \dots, g_{N_s} W^{N_s}\}$ generate $\mathcal{G}_x$. Express, for $1 \leq i \leq s$:

$$g_{N_i} = \sum_{\alpha \in (\mathbb{N})^h} x_1^{\alpha_1} \cdots x_h^{\alpha_h} a_\alpha^{(i)} \qquad a_\alpha \in k'[[x_{h+1}, \dots x_d]]. \tag{6.1.1}$$

We search for a finite set of coefficients, that span $\mathrm{Coeff}(\mathcal{G})_x$. A first candidate would be

$$\mathcal{F}_1' = \{a_\alpha^{(i)} W^{N_i - |\alpha|} / 0 \leq |\alpha| < N_i, 1 \leq i \leq s\}. \tag{6.1.2}$$

Consider the product of two elements in $\mathcal{F}$, say $g_{N_i} W^{N_i} \cdot g_{N_j} W^{N_j} = f_n W^n$ $(n = N_i + N_j)$; and a coefficient, say $a_{\alpha^{(1)}} W^{n-|\alpha^{(1)}|}$, of $f_n W^n$.

It follows from 6.1.1 that

$$a_{\alpha^{(1)}} = \sum_{\beta + \delta = \alpha^{(1)}} a_\beta^{(i)} a_\delta^{(j)},$$

for β, δ, and $\alpha^{(1)}$ in $(\mathbb{N})^h$. Note that we cannot extract from the previous, expressions of the form

$$a_{\alpha^{(1)}} W^{n-|\alpha^{(1)}|} = \sum_{\beta + \delta = \alpha^{(1)}} a_\beta^{(i)} W^{N_i - |\beta|} a_\delta^{(j)} W^{N_j - |\delta|}.$$

In fact, it can happen that $|\delta| \geq N_j$, and we only consider W with posive exponents. In particular, the previous expression of $a_{\alpha^{(1)}} W^{n-|\alpha^{(1)}|}$ is not weighted homogeneous in $\mathcal{F}_1'$, and hence not in the graded sub-algebra of $k'[[x_{h+1}, \dots x_d]][W]$ generated by $\mathcal{F}_1'$.

One way to remedy this situation is to allow $a_\beta^{(i)}$ to have weight $n - |\alpha^{(1)}|$ if $|\delta| \geq N_j$. Note that in such case

$$n - |\alpha^{(1)}| = N_i - |\beta| + N_j - |\delta| \leq N_i - |\beta|.$$

Therefore we enlarge $\mathcal{F}_1'$ to say,

$$\mathcal{F}_1 = \{a_\alpha^{(i)} W^{n_{i,\alpha}} / 0 \leq |\alpha| < N_i, 1 \leq i \leq s, 0 < n_{i,\alpha} \leq N_i - |\alpha|\}, \tag{6.1.3}$$

for N_i and α as in $\mathcal{F}_1$; and we can check that the coefficients of $f_n W^n$ are now weighted homogeneous on $\mathcal{F}_1$ (i.e. are in the sub-algebra of $k'[[x_{h+1}, \dots x_d]][W]$ generated by $\mathcal{F}_1$).

The argument applied here to $g_{N_i} W^{N_i} \cdot g_{N_j} W^{N_j}$, also holds for the coefficients of any product of elements in $\mathcal{F}$, and hence for the coefficients of any homogeneous element in the algebra generated by $\mathcal{F} = \{g_{N_1} W^{N_1}, \dots, g_{N_s} W^{N_s}\}$ (i.e. for the coefficients of any homogeneous element of $\mathcal{G}_x$).

This shows that there is an inclusion of subalgebras in $k'[[x_{h+1}, \dots x_d]][W]$, say

$$k'[[x_{h+1}, \dots x_d]][\mathcal{F}'_1] \subset \mathrm{Coeff}(\mathcal{G})_x \subset k'[[x_{h+1}, \dots x_d]][\mathcal{F}_1]. \tag{6.1.4}$$

On the other hand $k'[[x_{h+1}, \dots x_d]][\mathcal{F}'_1] \subset k'[[x_{h+1}, \dots x_d]][\mathcal{F}_1]$ is a finite extension (2.2, 2)). In particular $\mathrm{Coeff}(\mathcal{G})_x$ is finitely generated.

Remark 6.2. 1) $\mathcal{F}'_1$ can be extended to a finite set, say $\mathcal{F}''_1$, of generators of $\mathrm{Coeff}(\mathcal{G})_x$.

2) $Sing(k'[[x_{h+1}, \dots x_d]][\mathcal{F}'_1]) = Sing(\mathrm{Coeff}(\mathcal{G})_x)$ (Prop 4.4, (1)).

3) $Sing(\mathrm{Coeff}(\mathcal{G})_x)$ can be naturally identified with the intersection $Z \cap Sing(\mathcal{G})$ locally at the point x.

To check this last point 3) note first that the singular locus of $k'[[x_{h+1}, \dots x_d]][\mathcal{F}'_1]$ can be naturally identified with the intersection $Z \cap Sing(\mathcal{G})$. This follows from the definition of $\mathcal{F}'_1$ in (6.1.2), and the expressions in (6.1.1). Finally apply 2).

6.3. Fix, as in 6.1, an inclusion of smooth schemes $Z \subset V$, a closed point $x \in Z$. Assume that there is a retraction say $V \to Z$ locally at x. Let $x_1, \dots, x_h$ be part of a regular system of parameters at $\mathcal{O}_{V,x}$ so that $< x_1, \dots, x_h >$ defines Z at $\mathcal{O}_{V,x}$; and let $\{x_{h+1}, \dots x_d\}$ be a regular system of parameters at $\mathcal{O}_{Z,x}$. The local retraction at the point x defines an inclusion $\mathcal{O}_{Z,x} \subset \mathcal{O}_{V,x}$, so we may consider $\{x_1, \dots, x_h, x_{h+1}, \dots x_d\}$ as parameters at $\mathcal{O}_{V,x}$.

We may identify $\hat{\mathcal{O}}_{V,x}$ with a ring of formal power series $k'[[x_1, \dots, x_h, x_{h+1}, \dots x_d]]$, $\hat{\mathcal{O}}_{Z,x}$ with $k'[[x_{h+1}, \dots x_d]]$; and the local retraction defines the inclusion

$$k'[[x_{h+1}, \dots x_d]] \subset k'[[x_1, \dots, x_h, x_{h+1}, \dots x_d]].$$

Given $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_V[W])$, we have defined $\mathrm{Coeff}(\mathcal{G})$ at $\hat{\mathcal{O}}_{Z,x}[W]$. We now show that it can also be defined in $\mathcal{O}_{Z,x}[W]$, and that the definition relies on the local retraction and the local inclusion.

Express an element $f_n \in I_n \hat{\mathcal{O}}_{Z,x}$ as

$$f_n = \sum_{\alpha^{(1)} \in (\mathbb{N})^h} x_1^{\alpha_1} \cdots x_h^{\alpha_h} a_{\alpha^{(1)}},$$

$a_{\alpha^{(1)}} \in k'[[x_{h+1}, \dots x_d]]$. For each multi-index $\alpha^{(1)}$, $0 \leq |\alpha^{(1)}| \leq n$, the coefficient $a_{\alpha^{(1)}}$ can be identified with the class of $\Delta^{\alpha^{(1)}}(f_n)$ in $\hat{\mathcal{O}}_{Z,x}$. However, $\Delta^{\alpha^{(1)}}$ is a differential operator, relative to the local retraction $V \to Z$, $\Delta^{\alpha^{(1)}}(f_n)$ in an element in $\mathcal{O}_{V,x}$, and we can therefore consider the class of this element in $\mathcal{O}_{Z,x}$.

This shows that $\mathrm{Coeff}(\mathcal{G})(\subset \mathcal{O}_{Z,x}[W])$, is the restriction via $Z \subset V$, of the extension of $\mathcal{G}$ defined by the Diff-structure relative to the local retraction (3.6). In other words, and from an algebraic point of view, $\mathrm{Coeff}(\mathcal{G})(\subset \mathcal{O}_{Z,x}[W])$ is defined in terms of:
i) the surjection $\mathcal{O}_{V,x} \to \mathcal{O}_{Z,x}$; and
ii) the inclusion $\mathcal{O}_{Z,x} \subset \mathcal{O}_{V,x}$.

Lemma 6.4. *With the setting as above, the restriction of $G(\mathcal{G})$ to the smooth subscheme Z is the Diff-structure spanned by Coeff($\mathcal{G}$) (i.e. the Diff-structure generated by Coeff($\mathcal{G}$) in $\mathcal{O}_{Z,x}[W]$).*

Proof. The previous discussion shows that $\mathrm{Coeff}(\mathcal{G})$ is included in the restriction of $G(\mathcal{G})$, which is a Diff-structure over $\mathcal{O}_{Z,x}$ (Prop 5.1,A)). In particular, the Diff-structure spanned by $\mathrm{Coeff}(G)_x$ is included in the restriction. The claim is that this last inclusion is an equality.

Here $G(\mathcal{G}) = \bigoplus I'_k \cdot W^k$ is the Diff-structure generated by $\mathcal{G}$, so to prove this equality it suffices to show that given $f_n \in I_n$, and $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N})^d$, $0 \leq |\alpha| < n$, the class of $\Delta^{\alpha}(f_n)W^{n-|\alpha|}$ in $\mathcal{O}_{Z,x}[W]$, is in the Diff-structure generated by $\mathrm{Coeff}(\mathcal{G})$.

For this last claim we argue as in the proof of Prop 5.1, (A), by splitting each multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N})^d$:

$$\Delta^{\alpha} = \Delta^{\alpha^{(1)}} \Delta^{\alpha^{(2)}};$$

where $\alpha^{(1)} = (\alpha_1, \dots, \alpha_h)$, and $\alpha^{(2)} = (\alpha_{h+1}, \dots, \alpha_d)$.

The class of $\Delta^{\alpha^{(1)}}(f_n)W^{n-|\alpha^{(1)}|}$ is $a_{\alpha^{(1)}}W^{n-|\alpha^{(1)}|} \in \mathrm{Coeff}(\mathcal{G})$; and that of $\Delta^{\alpha}(f_n)W^{n-|\alpha|}$ is $\Delta^{\alpha^{(2)}}(a_{\alpha^{(1)}})W^{n-|\alpha^{(1)}|-|\alpha^{(2)}|}$, which is clearly in the Diff-structure spanned by $\mathrm{Coeff}(\mathcal{G})$.

Corollary 6.5. *Fix a smooth scheme V, a Rees algebra $\mathcal{G}$, and a smooth subscheme Z of V. If $G(\mathcal{G})$ denotes the Diff-structure spanned by $\mathcal{G}$, and if $[G(\mathcal{G})]_Z$ denotes the restriction to Z, then $Sing([G(\mathcal{G})]_Z)$, as closed set in Z, can be identified with $Z \cap Sing(\mathcal{G})$.*

This follows from Lemma 6.4 and 6.2, 3).

Remark 6.6. Let $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_{V'}[W])$ be a *Rees algebra* on a one dimensional smooth scheme V'. If we assume that some $I_k \neq 0$, then $Sing(\mathcal{G})$ is a finite set of points.

Fix $x \in Sing(\mathcal{G})$ and set $\hat{\mathcal{O}}_{V,x} = k'[[t]]$, so

$$\mathcal{G} = \bigoplus_{r \geq 1} < t^{a_r} > \cdot W^r,$$

and $a_r \geq r$ for each index r. Define

$$\lambda_{\mathcal{G}} = inf_r\{\frac{a_r}{r}\},$$

and note that $\lambda_{\mathcal{G}} \geq 1$.

Let $\{g_{N_1}W^{N_1}, \dots, g_{N_s}W^{N_s}\}$ be a set of generator locally at a closed point $x \in Sing(\mathcal{G})$. Fix any integer M divisible by all N_i, $1 \leq i \leq s$, then

$$\lambda_{\mathcal{G}} = \frac{\nu(I_M)}{M}$$

where $\nu(I_M)$ denotes the order of the ideal at $\mathcal{O}_{V',x}$. Let $\overline{\mathcal{G}}$ denote the integral closure of $\mathcal{G}$.

Claim 1: The integral closure of $\mathcal{G}$ is determined by the rational number $\lambda_{\mathcal{G}}$, and $\lambda_{\mathcal{G}} = \lambda_{\overline{\mathcal{G}}}$.

In fact, by usual arguments of toric geometry, we conclude that $t^n \cdot W^m \in \overline{\mathcal{G}}$, if and only if $\frac{n}{m} \geq \lambda_{\mathcal{G}}$. This proves the claim.

Let $G(\mathcal{G})$ denote the Diff-structure spanned by $\mathcal{G}$. Recall that $Sing(\mathcal{G}) = Sing(G(\mathcal{G}))$.

Claim 2: Locally at any $x \in Sing(\mathcal{G})$, both $\mathcal{G}$ and $G(\mathcal{G})$ have the same integral closure.

We prove our claim by showing that $\lambda_{\mathcal{G}} = \lambda_{\mathcal{G}'}$. To this end note that given $t^a \cdot W^b \in \mathcal{G}$, and an operator Δ^k, $0 \leq k < b$,

$$\Delta^k(t^a) \cdot W^{b-k} = d \cdot t^{a-k} \cdot W^{b-k},$$

where d is the class of an integer in the field k'. Since $a \geq b > k \geq 0$ it follows that $\frac{a-k}{b-k} \geq \frac{a}{b}$, so Claim 2 follows from Claim 1.

6.7. The previous Remark shows that in the one dimensional case, the extension $\mathcal{G} \subset G(\mathcal{G})$ is finite, where $G(\mathcal{G})$ is the Diff-structure spanned by $\mathcal{G}$.

In general $G(\mathcal{G})$ is not integrally closed. Consider, for example, the semi-group in $\mathbb{N} \times \mathbb{N}$ defined by the pairs (x, y) such that $-2x + y \geq 0$ and $-x + y \geq 3$. Use the previous remark to show that the set of pairs $\{t^i \cdot W^j\}$, where $((j, i)$ fulfills the previous inequalities, form a Diff-structure $\mathcal{G}$ which is not integrally closed. In fact $t^3 \cdot W$ is integral over $\mathcal{G}$.

6.8. Let $\mathcal{G}$ be a Rees algebra over a smooth scheme V, generated by elements $\{g_{N_1}W^{N_1}, \dots, g_{N_s}W^{N_s}\}$. Let M is a positive integer divisible by all N_j, $1 \leq j \leq s$; and consider the Rees ring $\mathcal{O}_V[I_M W^M]$. Note that $\mathcal{O}_V[I_M W^M] \subset \mathcal{G}$ is a finite extension of graded algebras, and therefore any Rees algebra is a finite extension of a Rees ring of an ideal (2.3).

Given two Rees algebras $\mathcal{G}_1 = \bigoplus_{r\geq 0} I(1)_r W^r$ and $\mathcal{G}_2 = \bigoplus_{r\geq 0} I(2)_r W^r$, there is a positive integer M such that both are integral extensions of the Rees ring generated by the M-th term, say $\bigoplus_{k\geq 0} I(1)^k_M W^{km}$ and $\bigoplus_{k\geq 0} I(2)^k_M W^{km}$.

Proposition 6.9. *Fix two Rees algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ on a smooth scheme V over a field k. Assume that for any morphism of regular k-schemes, say $V' \xrightarrow{\pi} V$, where V' is one dimensional, $\overline{\pi^{-1}(\mathcal{G}_1)} = \overline{\pi^{-1}(\mathcal{G}_2)}$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same integral closure in V.*

Proof. Choose M and ideals ideals $I(1)_M$ and $I(2)_M$ as in 6.8. We may assume here that π is of finite type. The previous properties show that under the condition of the hypothesis, both $I(1)_M$ and $I(2)_M$ have the same integral closure in $\mathcal{O}_V$ (2.5). In particular, $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same integral closure.

Proposition 6.10. *Let $\mathcal{G}_1 \subset \mathcal{G}_2(\subset \mathcal{O}_V[W])$ be a finite extension of Rees algebras on a smooth scheme V, and let V' be a smooth one dimensional subscheme in V. Fix $x \in V'$ and a regular system of coordinates $\{x_1, \dots, x_{d-1}, x_d\}$ at $\mathcal{O}_{V,x}$, so that the curve is locally defined by $< x_1, \dots, x_{d-1} >$. Then*

$$Coeff(\mathcal{G}_1) \subset Coeff(\mathcal{G}_2)$$

is a finite extension in $\mathcal{O}_{V'}[W]$.

Proof. Express any $f \in \hat{\mathcal{O}}_V = k'[[x_1, \dots, x_{d-1}, x_d]]$ as:

$$f = \sum_{\alpha \in (\mathbb{N})^{d-1}} x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} a_\alpha \quad a_\alpha \in k'[[x_d]].$$

The coefficients of fW^N are $\{a_\alpha W^{N-|\alpha|}/0 \leq |\alpha| < N\}$, and we define

$$sl_{V'}(fW^N) = min\{\frac{\nu(a_\alpha)}{N-|\alpha|}/0 \leq |\alpha| < N\},$$

where $\nu(a_\alpha)$ denotes the order of a_α in $k'[[x_d]]$. Set Coeff$(\mathcal{G}_1)$ and Coeff$(\mathcal{G}_2)$ in $\mathcal{O}_{V'}[W]$, as in (6.1). Assume that $\mathcal{F}_1 = \{f_{N_1}W^{N_1}, \dots, f_{N_s}W^{N_s}\}$ generate $\mathcal{G}_1$ locally at x, and that $\mathcal{F}_2 = \{g_{M_1}W^{M_1}, \dots, g_{M_t}W^{M_t}\}$ generate $\mathcal{G}_2$.

The inclusion Coeff$(\mathcal{G}_1) \subset$ Coeff$(\mathcal{G}_2)$ at $\mathcal{O}_{V'}[W]$ is clear.

Set Coeff$(\mathcal{G}_i) = \bigoplus_{r\geq 0} J(i)_r W^r$ in $\mathcal{O}_{V'}[W]$, for $i = 1, 2$. Note that $J(1)_r = 0$ for all $r \geq 1$ iff $V' \subset Sing(\mathcal{G}_1)$ iff $\mathcal{G}_1 \subset \bigoplus_{r\geq 0} P^r W^r$; where P is the ideal defining the smooth subscheme V'. Since $\mathcal{G}_1 \subset \mathcal{G}_2$ is finite, it follows that also $J(2)_r = 0$ for all $r \geq 1$.

Assume now that some $J(1)_r$ is not zero for some $r > 0$. The inclusion Coeff$(\mathcal{G}_1) \subset$ Coeff$(\mathcal{G}_2)$ ensures that

$$\lambda_{\text{Coeff}(\mathcal{G}_1)} \geq \lambda_{\text{Coeff}(\mathcal{G}_2)}, \tag{6.10.1}$$

and the claim is that they are equal (see Remark 6.6).

Each $g_{M_j}W^{M_j}$ is integral over the localization of $\mathcal{G}_1$ in $\mathcal{O}_{V,x}[W]$. And this property is preserved by any change of rings. Namely, for any ring homomorphism $\phi : \mathcal{O}_{V,x} \to S$, $\phi(\mathcal{G}_1)$ is a Rees algebra in $S[W]$, and $\phi(g_{M_j})W^{M_j}$ is integral over $\phi(\mathcal{G}_1)$.

Express, for any $g_{M_j}W^{M_j} \in \mathcal{F}_2$:

$$g_{M_j} = \sum_{\alpha \in (\mathbb{N})^h} x_1^{\alpha_1} \cdots x_h^{\alpha_h} a_\alpha^{(j)} \quad a_\alpha \in k'[[x_d]], \tag{6.10.2}$$

and set

$$\mathcal{F}'_2 = \{a_\alpha^{(j)} W^{M_j-|\alpha|}/0 \leq |\alpha| < M_j, 1 \leq j \leq t\}$$

(coefficients of all g_{M_j}'s).

We know that $k'[[x_d]][\mathcal{F}'_2] \subset$ Coeff$(\mathcal{G}_2)$ is a finite extension in $k'[[x_d]][W]$ (see 6.1.4); in particular:

$$\lambda_{\text{Coeff}(\mathcal{G}_2)} = min\{\frac{\nu(a_\alpha^{(j)})}{M_j - |\alpha|}/0 \leq |\alpha| < M_j, 1 \leq j \leq t\} \quad (6.6),$$

or, equivalently:

$$\lambda_{\text{Coeff}(\mathcal{G}_2)} = min\{sl_{V'}(g_{M_j}), 1 \leq j \leq t\}.$$

So equality in (6.10.1) would follow if we show that $\lambda_{\text{Coeff}(\mathcal{G}_1)} \leq \frac{\nu(a_\alpha^{(j)})}{M_j-|\alpha|}$ for each fraction as above.

We will assume that $\frac{\nu(a_{\alpha}^{(j_0)})}{M_{j_0}-|\alpha|} < \lambda_{\text{Coeff}(\mathcal{G}_1)}$ for some index $1 \leq j_0 \leq t$, or equivalently, that $sl_{V'}(g_{M_{j_0}}) < \lambda_{\text{Coeff}(\mathcal{G}_1)}$ for some index j_0, and show that in such case $g_{M_{j_0}}W^{M_{j_0}}$ is not integral over $\mathcal{G}_1$; which is a contradiction.

Define, as before, $sl_{V'}(f_{N_i})$ for each $f_{N_i}W^{N_i} \in \mathcal{F}_1$, so that

$$\lambda_{\text{Coeff}(\mathcal{G}_1)} = min\{sl_{V'}(f_{N_i}); 1 \leq i \leq s\}.$$

We claim that if $sl_{V'}(g_{M_{j_0}}) < \lambda_{\text{Coeff}(\mathcal{G}_1)}$, for some index j_0, a ring S and a morphism $\phi : \hat{\mathcal{O}}_V = k'[[x_1, \ldots, x_{d-1}, x_d]] \to S$ can be defined so that $\phi(g_{M_{j_0}})W^{M_{j_0}}$ is not integral over $\phi(\mathcal{G}_1)$.

Given $f = \sum_{\alpha} \lambda_{\alpha} x_1^{\alpha_1} \ldots x_{d-1}^{\alpha_{d-1}} x_d^{\alpha_d} \in k'[[x_1, \ldots, x_{d-1}, x_d]]$, set

$$Supp(f) = \{\alpha \in \mathbb{N}^d / \lambda_{\alpha} \neq 0\}.$$

Let $a > 0$ and $b > 0$ be positive integers such that

$$\lambda = \lambda_{\text{Coeff}(\mathcal{G}_1)} = \frac{a}{b}.$$

Define $l : \mathbb{R}^d \to \mathbb{R}$, $l(y_1, \ldots, y_d) = ay_1 + ay_2 + \cdots ay_{d-1} + by_d$, which maps $\mathbb{N}^d$ into $\mathbb{N}$.

For a fixed integer N:

$$l(N, 0, \ldots, 0, 0) = l(0, N, \ldots, 0, 0) = \cdots = l(0, \ldots, N, 0) = l(0, \ldots, 0, \lambda N) = aN.$$

Given $(\alpha_1, \ldots, \alpha_{d-1}, s) \in \mathbb{N}^d$, if $l(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, s) < aN$, $|\alpha| := \alpha_1 + \cdots + \alpha_{d-1} < N$. Furthermore:

$$l(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, s) < aN \Leftrightarrow a|\alpha| + bs < aN \Leftrightarrow \frac{s}{N - |\alpha|} < \lambda. \tag{6.10.3}$$

We show now that:

1) For each $f_{N_i}W^{N_i} \in \mathcal{F}_1$, $Supp(f_{N_i})$ is included in the half space $l(y_1, \ldots, y_d) \geq aN_i$.

2) For some $f_{N_i}W^{N_i} \in \mathcal{F}_1$, the intersection of $Supp(f_{N_i})$ with the hyperplane $l(y_1, \ldots, y_d) = aN_i$ is not empty.

3) For some $g_{M_{j_0}}W^{M_{j_0}} \in \mathcal{F}_2$, $Supp(g_{M_{j_0}})$ is not included in the half space $l(y_1, \ldots, y_d) \geq aM_{j_0}$.

In order to prove 1) set

$$f_{N_i} = \sum_{\alpha \in (\mathbb{N})^{d-1}} x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} a_{\alpha}^{(i)} \quad a_{\alpha}^{(i)} \in k'[[x_d]]. \tag{6.10.4}$$

and assume that

$$x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} x_d^s$$

is a monomial with non-zero coefficient in this expression (i.e. assume that $(\alpha_1, \ldots, \alpha_{d-1}, s) \in Supp(f_{N_i})$). The claim in 1) is that $l(\alpha_1, \ldots, \alpha_{d-1}, s) \geq aN_i$. In fact, if $l(\alpha_1, \ldots, \alpha_{d-1}, s) <$

aN_i, then $|\alpha| := \alpha_1 + \cdots + \alpha_{d-1} < N_i$ and $\frac{s}{N_i - |\alpha|} < \lambda$ (6.10.3). But in such case

$$sl_{V'}(f_{N_i}) \leq \frac{s}{N_i - |\alpha|} < \lambda = \lambda_{\mathrm{Coeff}(\mathcal{G}_1)} = min\{sl_{V'}(f_{N_i}); 1 \leq i \leq s\},$$

which is a contradiction.

Both conditions 2) and 3) follow similarly, from (6.10.3).

Set $S = k''[[t]]$ for some field extension k'' of k', and define $\beta : k'[[x_1, \dots, x_d]] \to k''[[t]]$ the continuous morphism, such that $\beta(x_i) = \lambda_i t^a$ ($\lambda_i \in k''$), for $1 \leq i \leq d-1$, and $\beta(x_d) = t^b$.

So $\beta(\mathcal{G}_1)$ is the Rees algebra in $k''[[t]][W]$ generated by $\{\beta(f_{N_i})W^{N_i}, 1 \leq i \leq s\}$.

We claim that for k'' an infinite field, and for sufficiently general $\lambda_i \in k''$:

1') $\beta(f_{N_i})$ has order at least aN_i in $k''[[t]]$.

2') $\beta(f_{N_{i_0}})$ has order aN_{i_0} for some $f_{N_{i_0}}W^{N_{i_0}} \in \mathcal{F}_1$.

3') $\beta(g_{M_{j_0}})$ has order strictly smaller then aM_{j_0}.

Finally Claim 1 in Remark 6.6, where now $\lambda_{\beta(\mathcal{G}_1)} = a$, asserts that $\beta(g_{M_{j_0}})W^{M_{j_0}}$ is not integral over $\beta(\mathcal{G}_1)$. $\square$

Remark 6.11. In the previous discussion we are given a smooth one dimensional subscheme V' in V, a point $x \in V'$; and we fixed a regular system of coordinates $\{x_1, \dots, x_{d-1}, x_d\}$ at $\mathcal{O}_{V,x}$, so that the curve is locally defined by $< x_1, \dots, x_{d-1} >$.

If $\mathcal{G}_1$ is locally generated by $\mathcal{F}_1 = \{f_{N_1}W^{N_1}, \dots, f_{N_s}W^{N_s}\}$, and each f_{N_i} has the formal expression (6.10.4), then Coeff($\mathcal{G}_1$) in $\hat{\mathcal{O}}_{V',x}[W]$ is defined, up to integral closure, by

$$\mathcal{F}'_1 = \{a^{(i)}_\alpha W^{N_i - |\alpha|} / 0 \leq |\alpha| < N_i, 1 \leq i \leq s\}.$$

It was indicated in 6.3, that Coeff($\mathcal{G}_1$)($\subset \mathcal{O}_{V',x}[W]$) was defined only in terms of:

i) the surjection $\mathcal{O}_{V,x} \to \mathcal{O}_{V',x}$; and

ii) the inclusion $\mathcal{O}_{V',x} \subset \mathcal{O}_{V,x}$.

Namely, in terms of the local inclusion $V' \subset V$, and the local (or formal) retraction $V \to V'$.

We claim now that in the case in which V' is one dimensional, and $x \in$ Sing ($\mathcal{G}_1$), then Coeff($\mathcal{G}_1$) is defined, up to integral closure, only in terms of the local inclusion $V' \subset V$ (i.e. only in terms of the surjection $\mathcal{O}_{V,x} \to \mathcal{O}_{V',x}$).

In our case the formal retraction is given by the inclusion $k[[x_d]] \subset k[[x_1, \dots, x_{d-1}, x_d]]$, and the inclusion by $< x_1, \dots, x_{d-1} >$. Fix x'_d so that $\{x_1, \dots, x_{d-1}, x'_d\}$ is also a regular system of parameters. Consider, on $\hat{\mathcal{O}}_{V',x}[W]$ the two Rees algebras, say Coeff($\mathcal{G}_1$) and Coeff($\mathcal{G}_1$)$'$, defined in terms parameters $\{x_1, \dots, x_{d-1}, x_d\}$ and $\{x_1, \dots, x_{d-1}, x'_d\}$ respectively. We claim that in the case $x \in$ Sing ($\mathcal{G}$), then $\lambda_{\mathrm{Coeff}(\mathcal{G})} = \lambda_{\mathrm{Coeff}(\mathcal{G})'}$ (see 6.6).

Set positive integers a, b, and $l : \mathbb{N}^d \to \mathbb{N}$, so that (1) and (2) in (6.10.3) hold for $\mathcal{F}_1$, where each f_{N_i} is defined in terms of the coordinates $\{x_1, \dots, x_{d-1}, x_d\}$ (6.10.4). And note that these conditions (1) and (2) imply that $\frac{a}{b} = \lambda_{\mathrm{Coeff}(\mathcal{G})}$.

Here we assume that $x \in Sing(\mathcal{G})$, and hence that $\lambda_{\mathrm{Coeff}(\mathcal{G})} = \frac{a}{b} \geq 1$. According to (2), for some $f_{N_i} \in \mathcal{F}$, there are monomials in $Supp(f_{N_i})$, say $x_1^{\alpha_1} \cdots x_{d-1}^{\alpha_{d-1}} x_d^s$, so that

$l(\alpha_1,\dots,\alpha_{d-1},s)=aN_i$. Among those monomials, choose one, say $x_1^{\beta_1}\cdots x_{d-1}^{\beta_{d-1}}x_d^r$, with the smallest value $\beta_1+\cdots+\beta_{d-1}$.

Both x_d and x'_d induce parameter of $\hat{\mathcal{O}}_{V',x}$ (via $\hat{\mathcal{O}}_{V,x}\to\hat{\mathcal{O}}_{V',x}$). After suitable change of parameters in $k[[x_d]]$ (which does not affect the definition of Coeff$(\mathcal{G})$):

$$x'_d=x_d+h_1x_1+h_2x_2+\cdots+h_{d-1}x_{d-1}.$$

As $a\geq b$, any monomial $x_1^{\alpha_1}\cdots x_{d-1}^{\alpha_{d-1}}x_d^s$, in $Supp(f_{N_i})$, gives rise to an expression

$$x_1^{\alpha_1}\cdots x_{d-1}^{\alpha_{d-1}}x_d^s=x_1^{\alpha_1}\cdots x_{d-1}^{\alpha_{d-1}}x_d'^s+\sum\gamma x_1^{\delta_1}\cdots x_{d-1}^{\delta_{d-1}}x_d'^j\in k[[x_1,\dots,x_{d-1},x'_d]],$$

with $l(\delta_1,\dots,\delta_{d-1},j)\geq aN_i$.

Furthermore, our choice of $x_1^{\beta_1}\cdots x_{d-1}^{\beta_{d-1}}x_d^r$, with the smallest value $\beta_1+\cdots+\beta_{d-1}$, shows that (1) and (2) also hold for the elements of $\mathcal{F}_1$, expressed now in the regular system of parameters $\{x_1,\dots,x_{d-1},x'_d\}$. So $\lambda_{\mathrm{Coeff}(\mathcal{G})}=\lambda_{\mathrm{Coeff}(\mathcal{G})'}$ as was to be shown.

The following Theorem can also be proved using Hironaka's theory on infinitely near points in [6]; a theory based on the behavior by monoidal transforms. Our proof relies on the previous development in this section, which will also be used for the proof of Theorem **??**.

Theorem 6.12. *Let $\mathcal{G}_1\subset\mathcal{G}_2$ be an inclusion of Rees algebras on a smooth scheme V. Let $G(\mathcal{G}_i)$ be the Diff-structure spanned by $\mathcal{G}_i$ $(i=1,2)$. If $\mathcal{G}_1\subset\mathcal{G}_2$ is a finite extension, then $G(\mathcal{G}_1)\subset G(\mathcal{G}_2)$ is a finite extension.*

Proof. The inclusion $G(\mathcal{G}_1)\subset G(\mathcal{G}_2)$ is clear. We will argue locally at a point $x\in Sing(\mathcal{G}_1)$, and we make use of the criterium in Proposition 6.9 to show that the extension is finite. Let $\mathcal{F}_1=\{f_{N_1}W^{N_1},\dots,f_{N_s}W^{N_s}\}$ generate $\mathcal{G}_1$ locally at x, and let $\mathcal{F}_2=\{g_{M_1}W^{M_1},\dots,g_{M_t}W^{M_t}\}$ generate $\mathcal{G}_2$.

Set $\pi:V'\to V$ where V' is one dimensional, and let $x'\in V'$ map to x. Locally at x', one can factor π as $V'\subset V''\to V$, so that $\phi:V''\to V$ is smooth. Let $\phi^{-1}(\mathcal{G}_1)$, $\phi^{-1}(\mathcal{G}_2)$ denote the total transforms of $\mathcal{G}_1$, $\mathcal{G}_2$; and $\phi^{-1}(G(\mathcal{G}_1))$, $\phi^{-1}(G(\mathcal{G}_2))$ be the total transforms of $G(\mathcal{G}_1)$, $G(\mathcal{G}_2)$.

If $\{x_1,\dots,x_d\}$ is a regular system of parameters at $\mathcal{O}_{V,x}$, then $\{x_1,\dots,x_d\}$ extends to a regular system of parameters, say $\{x_1,\dots,x_d,\cdots,x_e\}$ at $\mathcal{O}_{V'',x'}$. It is easy to check that

1) $\mathcal{F}_1=\{f_{N_1}W^{N_1},\dots,f_{N_s}W^{N_s}\}$ generate $\phi^{-1}(\mathcal{G}_1)$ locally at $\mathcal{O}_{V'',x'}$;
2) $\mathcal{F}_2=\{g_{M_1}W^{M_1},\dots,g_{M_t}W^{M_t}\}$ generate $\phi^{-1}(\mathcal{G}_2)$ at $\mathcal{O}_{V'',x'}$.
3) $\phi^{-1}(G(\mathcal{G}_1))$ is the Diff-structure generated by $\phi^{-1}(\mathcal{G}_1)$.
4) $\phi^{-1}(G(\mathcal{G}_2))$ is the Diff-structure generated by $\phi^{-1}(\mathcal{G}_2)$.

Therefore the setting at V and at V'' is the same, and hence, in order to apply Proposition 6.9 we need only to show that given a finite extension $\mathcal{G}_1\subset\mathcal{G}_2$, the *restrictions* of the Diff-structures $G(\mathcal{G}_i)$, $i=1,2$, to a smooth one dimensional scheme V', have the same integral closure.

Lemma 6.4 says that the restriction of $G(\mathcal{G}_i)$ to V' is the Diff-structure generated by Coeff$(\mathcal{G}_i)$ $(i=1,2)$. Remark 6.6 shows that for each index $i=1,2$, the Rees algebra Coeff$(\mathcal{G}_i)$, and the Diff-structure generated by Coeff$(\mathcal{G}_i)$, have the same integral closure. So it suffices

to show that Coeff($\mathcal{G}_1$) and Coeff($\mathcal{G}_2$) have the same integral closure, which was proved in Prop 6.10. In fact, 1),2),3), and 4) show that the setting of Prop 6.10 hold.

Theorem 6.13. *Let $\mathcal{G}_1 \subset \mathcal{G}_2$ be an inclusion of Rees algebras on a smooth scheme V. Fix a smooth subscheme $Z \subset V$, and a local (or formal) retraction $V \to Z$. If $\mathcal{G}_1 \subset \mathcal{G}_2$ is a finite extension, then Coeff($\mathcal{G}_1$) $\subset$ Coeff($\mathcal{G}_2$) is also finite.*

Theorem 6.14. *Let $\mathcal{G}$ be a Rees algebras on a smooth scheme V. Fix a point $x \in$ Sing ($\mathcal{G}$), a smooth subscheme $Z \subset V$ containing x, and two local (or formal) retractions, say $\pi : V \to Z$ and $\pi' : V \to Z$ at x. If Coeff($\mathcal{G}$) and Coeff($\mathcal{G}$)$'$ are defined in terms of π and π' respectively, then both have the same integral closure in $\mathcal{O}_{Z,x}[W]$.*

Proof. Both Theorems 6.13 and 6.14 can be treated similarly. As in the previous proof we apply the criterium in Proposition 6.9.

Set $\pi : V' \to Z$ where V' is smooth and one dimensional, and let $x' \in V'$ map to x. Locally at x', one can factor π as $V' \subset Z_1 \to Z$, so that $\phi : Z_1 \to Z$ is smooth. A retraction of V on Z can be lifted to the fiber product, say $V_1 \to Z_1$, and the construction of Coeff is compatible with base change. By further restriction to V' we may assume that Z_1 is one dimensional. Theorem 6.13 follows now from Prop 6.10, and Theorem 6.14 from 6.11

□

7. Further applications.

There is a particular but natural morphism among smooth schemes, namely that defined by blowing up closed and smooth centers (i.e. monoidal transformations). Given an ideal in a smooth scheme, there are several notions of transformations of sheaves of ideals defined in terms of monoidal transformations (e.g. total transforms, weak transforms, and strict transforms of ideals.).

Questions as resolution of singularities, or Log principalization of ideals, are formulated in terms of these notions of transformations. In the case of schemes over fields of characteristic zero, both resolution and Log principalization of ideals are two well known theorems due to Hironaka. If two ideals have the same integral closure, then a Log-principalization of one of them is also a Log-principalization of the other; the key point being that the transforms of both ideals also have the same integral closure.

Notions of transformations on ideals extend naturally to graded structures of ideals. And again, if two graded structures have the same integral closure, then their transforms are graded rings with the same integral closure.

Both theorems of Log-principalization of ideals and resolution of singularities are proved by induction on the dimesion of the ambient space. In the setting of differential structures this form of induction relates to the notion of restriction on a smooth subschemes, say $Z \subset V$ in Theorems 6.13.

The outcome of Theorems 6.14 is that such form of restriction on Z is, up to integral closure, independent of the particular retraction. This result plays a role in the extension

of resolution theorems to graded structures. We refer here to [6] or [7] for the notions of trasformations of differentiable structures, and related results.

Our development will be applied in [16], in relation with the study of hypersurface singularities over fields of positive characteristic.

References

[1] M. Fernández-Lebrón and Luis Narváez-Macarro, 'Hesse-Schmidt Derivations and Coefficient Fields in Positive Characteristics', *J. Algebra*, 265 (2003), no. 1, 200–210.

[2] J. Giraud, 'Sur la theorie du contact maximal', *Math. Zeit.*, 137 (1972), 285-310.

[3] J. Giraud. 'Contact maximal en caractéristique positive', *Ann. Scien. de l'Ec. Norm. Sup. 4ème série*, 8 (1975) 201-234.

[4] A. Grothendieck and J. Dieudonné, 'Éléments de Géométrie Algébrique IV:Étude locale des schémas (Quatrième Partie), vol. 32. Inst. Hautes Études Sci. Publ. Math., Press Univ. de France, Paris, (1967).

[5] H. Hironaka, 'Idealistic exponent of a singularity', *Algebraic Geometry*, The John Hopkins centennial lectures, Baltimore, John Hopkins University Press (1977), 52-125.

[6] H. Hironaka, 'Theory of infinitely near singular points', Journal Korean Math. Soc. 40 (2003), No.5, pp. 901-920

[7] H. Hironaka, 'Three key theorems on infinitely near singularities ', Singularits franco-japonaises Jean-Paul Brasselet - Tatsuo Suwa (d.) Sminaires et Congrs 10 SOCIETE MATHEMATIQUE DE FRANCE - SMF (2005)

[8] J. Kollár, 'Resolution of Singularities- Seattle Lecture' arXiv:math.AG/ 0508332 v1 17 Aug 2005.

[9] J. Lipman and B. Teissier , 'Pseudo-rational local rings and a theorem of Briancon-Skoda about integral closures of ideals'. Michigan Math. J. (1981) 28, pp 97-116.

[10] H. Matsumura, *Commutative Algebra (second Edition)* Mathematics Lecture Note Series, Benjamin, Cumming Publishing Company (1980).

[11] K. R. Mount and O. Villamayor (Sr) *Taylor Series and higher derivations.* Vol 18, Impresiones del Departamento de Matematicas, Universidad de Buenos Aires (1968, reimpresion 1979).

[12] L. Narváez Macarro *A note on the behaviour under a ground field extension of quasicoefficient fields* J. London Math. Soc. (2) 43 (1991) 12-22.

[13] W. Traves 'Localization of the Hasse-Schmidt algebra', Canad. Math. Bull. 46 (2003), no. 2, 304–309.

[14] W. V. Vasconcelos *Arithmetic of Blowup Algebras* London Mathematical Society. Lecture Notes Series 195. Cambridge University Press 1994.

[15] O. Villamayor, *Differential operators on smooth schemes and embedded singularities.* Preprint.

[16] O. Villamayor, *Differential operators and singularities in positive characteristic.* In preparation.

[17] J. Wlodarczyk, 'Simple Hironaka resolution in characteristic zero', to appear (2005).

[18] A. Yakutieli, 'An explicit construction of the Grothendieck residual complex', Asterisque, vol. 208, Soc. Math. France, (1992).

[19] B. Youssin, 'Newton Polyhedra without coordinates', Mem. Amer. Math. Soc. 433 (1990), 1-74,75-99.

[20] O. Zariski, P. Samuel *Commutative Algebra (Vol II)* University Series in Higher Mathematics. D. Van Nostrand Company (1960).

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